

Stabilizing Networked Control Systems with Sparse Observer-Controller Networks

Mohammad Razeghi-Jahromi and Alireza Seyedi, *Senior Member, IEEE*

Abstract

In this paper we provide a set of distributed stability conditions for linear time-invariant networked control systems with arbitrary topology, using a Lyapunov direct approach. We then use these stability conditions to provide a novel approach for the design of a sparse, distributed, observer-based control network. The sparsity of the control network is motivated by the desire to minimize its cost and complexity. We employ distributed observers by employing the output of other nodes to improve the stability of each observer dynamics. To avoid unbounded growth of local controller and observer gains, we impose bounds on the norm of controller and observer gains. The effects of relaxation of these bounds is discussed.

Index Terms

Networked control systems, Distributed observer-based control, Linear time-invariant systems, Sparse control network

I. INTRODUCTION

Control systems with spatially distributed components have been in use for several decades. In early systems, the components were connected via dedicated hard-wired connections which carried the information from the sensors to a central location, where control signals were computed and sent to the actuators. Today, advances in computing and communications technology have provided us with low cost and low power processing capability and have enabled us to exchange information via efficient communication networks. These advances have considerably widened the scope of the research on spatially distributed control systems to include communications and network effects explicitly, as they significantly affect the dynamic behavior of the entire system.

Spatially distributed control systems can be abstracted within the framework of networked control systems (NCS). An NCS consists of a number of subsystems, each comprising of a plant and a controller, coupled together in a network structure. The interaction of plants with each other forms the *plant network*. Control signals are exchanged using the *control network*¹ (Fig. 1). Networked control systems have a wide range of applications including electrical power networks [1], transportation networks [2], factory automation [3], tele-operations [4] and sensor and actuator networks [5].

Within this framework, a centralized control approach can be modeled by considering a complete control network, which provides all controllers with access to the states of all plants. However, in general, it is not practical to control a large-scale networked system with the centralized approach, where the control law uses the state information of all subsystems, as this requires a large and costly control network for exchanging state information. To overcome this limitation, we must resort to decentralized or distributed control strategies [6]-[9]. The decentralized control strategy lies at the opposite end of the spectrum from the centralized approach, where the control law uses only a subsystem's local state information to control the given subsystem. In other words there is no control network. Such local controls can be effective when the couplings between subsystems are weak [10]-[15]. However, when the coupling between subsystems are not weak, we may have to use a distributed control approach, where each subsystem uses its own state as well as the state of some other subsystems. This is a middle-of-the-road solution, between centralized and decentralized approaches. Hence, it can achieve asymptotic stability given stronger subsystem coupling, compared to the decentralized control strategy [16],[17], yet avoid the complexity and cost of a centralized approach.

Whether a centralized, distributed or decentralized approach is taken, both the dynamics of each subsystem and the network topology, play important roles in the stability of the overall network. It is easy to verify that even if each subsystem is asymptotically stable in isolation, the connected plants may be unstable due to the interactions between them. In such a scenario to stabilize the NCS a control network carrying state or feedback information between different subsystems may be necessary.

A. Related Literature

In general, the networked control literature can be classified into two main groups. The first group focuses on the effects of the impairments and limitations imposed by closing a feedback loop through a communication channel, including bandwidth

M. Razeghi-Jahromi is with the Department of Electrical and Computer Engineering, University of Rochester, Rochester, NY (e-mail: jahromi@ece.rochester.edu). A. Seyedi is with the Department of Electrical Engineering and Computer Science, University of Central Florida, Orlando, FL (e-mail: alireza.seyedi@ieee.org).

¹Also known as information network, communications network, or feedback network.

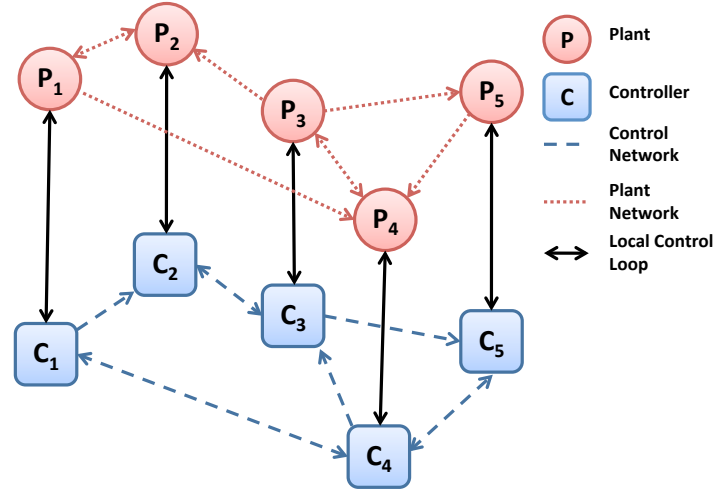


Fig. 1. A Networked Control System (NCS)

constraint, packet dropout, sampling and delay [18]–[27]. The second group, in which this work should be placed, considers the topological and network effects, and investigates how the topology of the plant network affects the overall network behavior, and how a control network can be designed that results in stability or desired performance.

For both decentralized and distributed control approaches, existing works have studied the problem of imposing a priori constraints on communication requirements between subsystems. It has been shown that under a structural condition, namely *quadratic invariance*, finding optimal controllers can be cast as a convex optimization problem [28]–[32]. Other work have shown similar results, conditioned on the network being a partially ordered set (poset) [33]–[35]. This constraint is closely related to quadratic invariance, however, it can lead to more computationally efficient solutions and provides better insight into the topology of the optimal controllers.

While these results are both elegant and important, they impose restrictions on the topology of the plant network. For networks with arbitrary topology, the key question concerning the design of the control network is one of *topological information requirements* and can be framed as: *Which nodes should be given the state and output information of a particular node, in order for the local controllers to be able to satisfy a global control objective?* This question is critical in the design of massively distributed control systems, such as the Smart Grid [1],[36]–[39]. In addressing this key question, the goal is often to minimize the cost and complexity of the control network. These goals take the form of sparsity constraints or objective functions, assuming that the cost of a link is fixed. This problem has been considered in various settings and solutions have been proposed that can be used to find suboptimal controllers [40]–[42].

B. Our Contribution

In this paper, we first develop a set of distributed stability conditions that guarantee global asymptotic stability, using the Lyapunov direct method. These conditions extend our prior results in [43] by inclusion of state estimation, among other improvements. We then use these conditions to explore the problem of designing a sparse observer-controller network for a given plant network with arbitrary topology. We therefore take a broader look at the topological information requirements by taking into account the distributed state estimation problem, generally neglected by the existing works. We proceed to provide a solution for finding the sparsest observer-controller network that satisfies our set of stability conditions.

Our setting is based on a linear time-invariant (LTI) NCS with arbitrary topology. We assume that each local controller includes a state observer to estimate the local states. We design distributed observers and controllers that use outputs and estimated states from potentially all other subsystems to improve the stability of observer dynamics and provide the control signals to the subsystems. We assume the controllers communicate with each other through a *perfect* control network to exchange estimated state and control signal information. By perfect, we mean that communication links do not have any bandwidth limitation, data loss or induced network delays. We assume that the cost of all links are identical, regardless of how much information they carry. Therefore, the goal of reducing the cost of the control network is equivalent to minimizing the number of its links while guaranteeing global stability.

The remaining of this paper is organized as follows. Section II establishes our notation and describes the networked control system under consideration. Section III derives stability conditions that guarantees global asymptotic stability of the overall network. In Section IV we design the control network with minimum necessary number of links that satisfy the network stability conditions. In Section V we demonstrate our proposed approach using numerical examples. Concluding remarks are given in Section VI.

II. NOTATION AND PROBLEM DEFINITION

1) *Notation:* We use \mathbb{R}_+ and \mathbb{R}_{++} to denote the sets of non-negative real and positive real numbers, respectively. The set of real n -vectors is denoted by \mathbb{R}^n , the set of real $m \times n$ matrices is denoted by $\mathbb{R}^{m \times n}$, and the set of real symmetric positive definite $n \times n$ matrices is denoted by \mathbb{S}_{++}^n . Matrices and vectors are denoted by capital and lower-case bold letters, respectively. The symbol \preceq (and its strict form \prec) is used to denote generalized matrix inequality defined by the positive definite cone between symmetric matrices. The Euclidean (l_2) vector norm is represented by $\|\cdot\|$. When applied to a matrix $\|\cdot\|$ denotes the induced l_2 norm, or spectral matrix norm: $\|\mathbf{A}\|^2 = \lambda_{\max}(\mathbf{A}^T \mathbf{A}) = \sigma_{\max}^2(\mathbf{A})$. By $\lambda_{\min}(\mathbf{B})$, $\lambda_{\max}(\mathbf{B})$ and $\sigma_{\max}(\mathbf{A})$ we denote the smallest and largest eigenvalues of symmetric matrix \mathbf{B} and the largest singular value of matrix \mathbf{A} , respectively. $\mathbf{I}_n \in \mathbb{R}^{n \times n}$ is the identity matrix. We let \mathcal{N} denote the set $\{1, 2, \dots, N\}$. The cardinality of a set is denoted by $|\cdot|$ and the indicator function of x is represented by 1_x .

2) *Problem Definition:* The system under study is a collection of N coupled linear time-invariant subsystems, each comprising of a plant and a controller. The state of the i th plant, which may be affected by all other subsystems, is a function $\mathbf{x}_i(t) : \mathbb{R}_+ \rightarrow \mathbb{R}^{n_i}$, governed by

$$\begin{aligned}\dot{\mathbf{x}}_i(t) &= \mathbf{A}_i \mathbf{x}_i(t) + \mathbf{B}_i \mathbf{u}_i(t) + \sum_{j \in \mathcal{N}_i} \mathbf{H}_{ij} \mathbf{x}_j(t), \\ \mathbf{y}_i(t) &= \mathbf{C}_i \mathbf{x}_i(t),\end{aligned}\tag{1}$$

where $\mathcal{N}_i \triangleq \mathcal{N} - \{i\}$ and n_i is the state space dimension of subsystem i . The signal $\mathbf{u}_i(t) : \mathbb{R}_+ \rightarrow \mathbb{R}^{m_i}$ is the control signal generated by the i th controller, where m_i is the dimension of the control set and $\mathbf{y}_i(t) : \mathbb{R}_+ \rightarrow \mathbb{R}^{r_i}$ is the output of subsystem i with dimension r_i . \mathbf{A}_i , \mathbf{B}_i , \mathbf{C}_i and \mathbf{H}_{ij} are constant known matrices with compatible size. We consider an arbitrary directed network. That is, \mathbf{H}_{ij} and \mathbf{H}_{ji} are not necessarily equal. We assume that $(\mathbf{A}_i, \mathbf{B}_i)$ is fully controllable and $(\mathbf{A}_i, \mathbf{C}_i)$ is fully observable. We look for distributed stabilizing observer-based controller of the form

$$\begin{aligned}\dot{\hat{\mathbf{x}}}_i(t) &= \mathbf{A}_i \hat{\mathbf{x}}_i(t) + \mathbf{B}_i \mathbf{u}_i(t) + \sum_{j \in \mathcal{N}_i} \mathbf{H}_{ij} \hat{\mathbf{x}}_j(t) + \mathbf{M}_i (\mathbf{C}_i \hat{\mathbf{x}}_i(t) - \mathbf{y}_i(t)) + \sum_{j \in \mathcal{N}_i} \mathbf{O}_{ij} (\mathbf{C}_j \hat{\mathbf{x}}_j(t) - \mathbf{y}_j(t)), \\ \mathbf{u}_i(t) &= \mathbf{K}_i \hat{\mathbf{x}}_i(t) + \sum_{j \in \mathcal{N}_i} \mathbf{L}_{ij} \hat{\mathbf{x}}_j(t),\end{aligned}\tag{2}$$

where $\hat{\mathbf{x}}_i(t) : \mathbb{R}_+ \rightarrow \mathbb{R}^{n_i}$ is the estimate of $\mathbf{x}_i(t)$, \mathbf{K}_i and \mathbf{L}_{ij} are local and coupling controller gains, and \mathbf{M}_i and \mathbf{O}_{ij} are local and coupling observer gains, respectively. Note that to estimate $\mathbf{x}_i(t)$, we not only use output of subsystem i , but also outputs of (potentially) all other subsystems. This is dual to the concept of distributed control.

We also consider constraints

$$\|\mathbf{K}_i\| \leq \kappa_i, \quad \|\mathbf{M}_i\| \leq \mu_i, \quad \|\mathbf{L}_{ij}\| \leq \iota_{ij}, \quad \|\mathbf{O}_{ij}\| \leq \omega_{ij},\tag{3}$$

to avoid undesirably large gains. Later (see Section IV), we will discuss the effect of relaxation of these constraints.

Our objective is to find distributed observer-based control law (2), using feedback from (potentially) all other subsystems to stabilize the plant network with a sparse control network. That is, we need to find \mathbf{K}_i , \mathbf{M}_i , \mathbf{L}_{ij} and \mathbf{O}_{ij} , such that the overall network is globally asymptotically stable and that the number of links in the control network (number of non-zero coupling gains \mathbf{L}_{ij} and \mathbf{O}_{ij}) is minimized.

By defining error signal $\mathbf{e}_i(t) \triangleq \hat{\mathbf{x}}_i(t) - \mathbf{x}_i(t)$ and substituting $\hat{\mathbf{x}}_i(t)$ in (1) and (2), the controlled system reduces to

$$\dot{\mathbf{x}}_i(t) = (\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i) \mathbf{x}_i(t) + \sum_{j \in \mathcal{N}_i} (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij}) \mathbf{x}_j(t) + \mathbf{B}_i \mathbf{K}_i \mathbf{e}_i(t) + \sum_{j \in \mathcal{N}_i} \mathbf{B}_i \mathbf{L}_{ij} \mathbf{e}_j(t),\tag{4}$$

$$\dot{\mathbf{e}}_i(t) = (\mathbf{A}_i + \mathbf{M}_i \mathbf{C}_i) \mathbf{e}_i(t) + \sum_{j \in \mathcal{N}_i} (\mathbf{O}_{ij} \mathbf{C}_j + \mathbf{H}_{ij}) \mathbf{e}_j(t).\tag{5}$$

This is a networked linear cascade dynamical system with the equilibrium point $(\mathbf{x}, \mathbf{e}) = (\mathbf{0}, \mathbf{0})$, where $\mathbf{x} \triangleq [\mathbf{x}_1^T \quad \mathbf{x}_2^T \quad \dots \quad \mathbf{x}_N^T]^T$ and $\mathbf{e} \triangleq [\mathbf{e}_1^T \quad \mathbf{e}_2^T \quad \dots \quad \mathbf{e}_N^T]^T$.

III. NETWORK STABILITY CONDITIONS

In this section we derive conditions that assure the entire network is globally asymptotically stable. To do this, we will use the following lemma to find network stability conditions that guarantee (4) and (5) are globally asymptotically stable.

Lemma 1 [44], [45]: If the system (4), with $\mathbf{e}(t)$ viewed as the input, is input-to-state stable and the equilibrium point of (5), which is $\mathbf{e} = \mathbf{0}$, is globally asymptotically stable, then the equilibrium point of the networked linear cascade dynamical system (4) and (5), which is $(\mathbf{x}, \mathbf{e}) = (\mathbf{0}, \mathbf{0})$, is globally asymptotically stable.

Based on this lemma, we provide the following theorems:

Theorem 1: System (4) is input-to-state stable if there exist \mathbf{K}_i , \mathbf{L}_{ij} , \mathbf{P}_i and Δ_{ij} such that

$$(\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i)^T \mathbf{P}_i + \mathbf{P}_i (\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i) + \sum_{j \in \mathcal{N}_i} [\mathbf{P}_i (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij}) \Delta_{ji}^{-1} (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})^T \mathbf{P}_i + \Delta_{ij}] \prec \mathbf{0},$$

$$\mathbf{P}_i, \Delta_{ik} \succ \mathbf{0}, \quad (6)$$

for all $i, k \in \mathcal{N}$.

Proof: Consider the quadratic Lyapunov function for each isolated subsystem i as

$$V_i(\mathbf{x}_i) = \mathbf{x}_i^T \mathbf{P}_i \mathbf{x}_i, \quad (7)$$

where $\mathbf{P}_i \in \mathbb{S}_{++}^{n_i}$. Then, a good quadratic Lyapunov function candidate for the entire network, $V : \mathbb{R}^{\sum_i n_i} \rightarrow \mathbb{R}_+$, is

$$V(\mathbf{x}) = \sum_{i \in \mathcal{N}} V_i(\mathbf{x}_i) = \sum_{i \in \mathcal{N}} \mathbf{x}_i^T \mathbf{P}_i \mathbf{x}_i. \quad (8)$$

Taking derivative of (7) along the trajectories of unforced system (4), we have

$$\begin{aligned} \dot{V}_i(\mathbf{x}_i) &= \dot{\mathbf{x}}_i^T \mathbf{P}_i \mathbf{x}_i + \mathbf{x}_i^T \mathbf{P}_i \dot{\mathbf{x}}_i \\ &= \mathbf{x}_i^T [(\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i)^T \mathbf{P}_i + \mathbf{P}_i (\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i)] \mathbf{x}_i + 2 \sum_{j \in \mathcal{N}_i} \mathbf{x}_i^T \mathbf{P}_i (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij}) \mathbf{x}_j. \end{aligned} \quad (9)$$

To upper bound the cross terms in (9), we use the following inequality which is a special case of Fenchel's inequality [46]: For any $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{R} \in \mathbb{R}^{n \times m}$ and $\Delta \in \mathbb{S}_{++}^n$ we have

$$2\mathbf{z}^T \mathbf{R} \mathbf{y} \leq \mathbf{z}^T \Delta \mathbf{z} + \mathbf{y}^T \mathbf{R}^T \Delta^{-1} \mathbf{R} \mathbf{y}. \quad (10)$$

Hence, (9) can be upper bounded, using (10), as

$$\begin{aligned} \dot{V}_i(\mathbf{x}_i) &\leq \mathbf{x}_i^T [(\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i)^T \mathbf{P}_i + \mathbf{P}_i (\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i)] \mathbf{x}_i \\ &\quad + \sum_{j \in \mathcal{N}_i} [\mathbf{x}_i^T \mathbf{P}_i (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij}) \Delta_{ji}^{-1} (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})^T \mathbf{P}_i \mathbf{x}_i + \mathbf{x}_j^T \Delta_{ji} \mathbf{x}_j]. \end{aligned} \quad (11)$$

Therefore, for the entire system we have

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \sum_{i \in \mathcal{N}} \dot{V}_i(\mathbf{x}_i) \\ &\leq \sum_{i \in \mathcal{N}} \mathbf{x}_i^T [(\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i)^T \mathbf{P}_i + \mathbf{P}_i (\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i)] \mathbf{x}_i \\ &\quad + \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i} [\mathbf{x}_i^T \mathbf{P}_i (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij}) \Delta_{ji}^{-1} (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})^T \mathbf{P}_i \mathbf{x}_i + \mathbf{x}_j^T \Delta_{ji} \mathbf{x}_j]. \end{aligned} \quad (12)$$

Using the following reorganization of the terms

$$\sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i} \mathbf{x}_j^T \Delta_{ji} \mathbf{x}_j = \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i} \mathbf{x}_i^T \Delta_{ij} \mathbf{x}_i, \quad (13)$$

(12) can be written as

$$\begin{aligned} \dot{V}(\mathbf{x}) &\leq \sum_{i \in \mathcal{N}} \mathbf{x}_i^T [(\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i)^T \mathbf{P}_i + \mathbf{P}_i (\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i) \\ &\quad + \sum_{j \in \mathcal{N}_i} (\mathbf{P}_i (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij}) \Delta_{ji}^{-1} (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})^T \mathbf{P}_i + \Delta_{ij})] \mathbf{x}_i. \end{aligned} \quad (14)$$

To ensure $\dot{V}(\mathbf{x}) < 0$, it suffices for its upper bound in (14) to be negative. Therefore, (6) is a sufficient condition that guarantees global asymptotic stability of the equilibrium point $\mathbf{x} = \mathbf{0}$ of unforced system (4). In other words, if (6) holds, the system (4) is input-to-state stable. \blacksquare

Theorem 2: The equilibrium point, $\mathbf{e} = \mathbf{0}$, of (5) is globally asymptotically stable, if there exist \mathbf{M}_i , \mathbf{O}_{ij} , $\hat{\mathbf{P}}_i$ and $\hat{\Delta}_{ij}$ such that

$$(\mathbf{A}_i + \mathbf{M}_i \mathbf{C}_i)^T \hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i (\mathbf{A}_i + \mathbf{M}_i \mathbf{C}_i) + \sum_{j \in \mathcal{N}_i} [(\mathbf{O}_{ji} \mathbf{C}_i + \mathbf{H}_{ji})^T \hat{\Delta}_{ji}^{-1} (\mathbf{O}_{ji} \mathbf{C}_i + \mathbf{H}_{ji}) + \hat{\mathbf{P}}_i \hat{\Delta}_{ij} \hat{\mathbf{P}}_i] \prec \mathbf{0},$$

$$\hat{\mathbf{P}}_i, \hat{\Delta}_{ik} \succ \mathbf{0}, \quad (15)$$

for all $i, k \in \mathcal{N}$.

Proof: Dual of the proof of Theorem 1. ■

Corollary 1: The control network can be limited to be a subset of the plant network with no loss in its sparsity.

Proof: The terms involving \mathbf{H}_{ij} in (6) and (15) are positive semidefinite. Thus, for any i, j where $\mathbf{H}_{ij} = \mathbf{0}$, the best margin to satisfy (6) and (15) can be achieved by choosing $\mathbf{L}_{ij} = \mathbf{O}_{ij} = \mathbf{0}$, which is equivalent to completely decoupling the link. In other words, if there is a solution with non-zero \mathbf{L}_{ij} and \mathbf{O}_{ij} , where $\mathbf{H}_{ij} = \mathbf{0}$, a sparser solution can be found by setting $\mathbf{L}_{ij} = \mathbf{O}_{ij} = \mathbf{0}$. Thus, the control network need not have a link where there is no link in the plant network, and the sparsest control network will always be a subset of the plant network. ■

We can incorporate Corollary 1 into our network stability criteria by defining $\mathcal{N}_i^{\text{in}} \triangleq \{j \mid j \neq i, \mathbf{H}_{ij} \neq \mathbf{0}\}$ as the set of neighbors that affect subsystem i and $\mathcal{N}_i^{\text{out}} \triangleq \{j \mid j \neq i, \mathbf{H}_{ji} \neq \mathbf{0}\}$ as the set of neighbors that are affected by subsystem i . By replacing \mathcal{N}_i in equations (1), (2), (4) and (5) with $\mathcal{N}_i^{\text{in}}$ which changes the right hand side of equation (13) with $\mathcal{N}_i^{\text{out}}$, then our network stability criteria (6) and (15) become

$$(\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i)^T \mathbf{P}_i + \mathbf{P}_i (\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i) + \sum_{j \in \mathcal{N}_i^{\text{in}}} \mathbf{P}_i (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij}) \Delta_{ji}^{-1} (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})^T \mathbf{P}_i + \sum_{j \in \mathcal{N}_i^{\text{out}}} \Delta_{ij} \prec \mathbf{0},$$

$$\mathbf{P}_i, \Delta_{ik} \succ \mathbf{0}, \quad (16)$$

and

$$(\mathbf{A}_i + \mathbf{M}_i \mathbf{C}_i)^T \hat{\mathbf{P}}_i + \hat{\mathbf{P}}_i (\mathbf{A}_i + \mathbf{M}_i \mathbf{C}_i) + \sum_{j \in \mathcal{N}_i^{\text{out}}} (\mathbf{O}_{ji} \mathbf{C}_i + \mathbf{H}_{ji})^T \hat{\Delta}_{ji}^{-1} (\mathbf{O}_{ji} \mathbf{C}_i + \mathbf{H}_{ji}) + \sum_{j \in \mathcal{N}_i^{\text{in}}} \hat{\mathbf{P}}_i \hat{\Delta}_{ij} \hat{\mathbf{P}}_i \prec \mathbf{0},$$

$$\hat{\mathbf{P}}_i, \hat{\Delta}_{ik} \succ \mathbf{0}, \quad (17)$$

for all $i, k \in \mathcal{N}$ respectively. Recall that in general $\mathcal{N}_i^{\text{in}} \neq \mathcal{N}_i^{\text{out}}$, which means that neighborhoods is not necessarily a symmetric relation (i.e. the network is directed).

Theorem 3: The equilibrium point of the networked linear cascade dynamical systems (4) and (5), which is $(\mathbf{x}, \mathbf{e}) = (\mathbf{0}, \mathbf{0})$, is globally asymptotically stable if (16) and (17) are satisfied for all $i, k \in \mathcal{N}$.

Proof: Follows directly from Lemma 1, Theorem 1, Theorem 2 and Corollary 1. ■

Corollary 2: It is always possible to stabilize the overall network, neglecting the constraints on local controller and observer gains, if \mathbf{L}_{ij} and \mathbf{O}_{ij} exist that satisfy the matching conditions $\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij} = \mathbf{0}$ and $\mathbf{O}_{ij} \mathbf{C}_j + \mathbf{H}_{ij} = \mathbf{0}$.

Proof: Since the matching condition is satisfied for all links, the subsystem's dynamics in (4) and (5) are completely decoupled from their neighbors. Using controllability of $(\mathbf{A}_i, \mathbf{B}_i)$ and observability of $(\mathbf{A}_i, \mathbf{C}_i)$, we can find local feedback gains \mathbf{K}_i and \mathbf{M}_i such that $\mathbf{A}_i + \mathbf{B}_i \mathbf{K}_i$ and $\mathbf{A}_i + \mathbf{M}_i \mathbf{C}_i$ have arbitrary desired eigenvalues in the open left-half complex plane. This means that a control network identical to the plant network will stabilize the system, though it may not be sparse. ■

Using Theorem 3, our objective is now to satisfy the set of inequalities (16) and (17) with given bounds in (3). This is a feasibility problem in the sense that we need to find $\mathbf{P}_i, \hat{\mathbf{P}}_i, \mathbf{K}_i, \mathbf{M}_i, \mathbf{L}_{ij}, \mathbf{O}_{ij}, \Delta_{ij}$ and $\hat{\Delta}_{ij}$ such that (3), (16) and (17) are satisfied.

IV. SPARSE CONTROL NETWORK DESIGN

As discussed in Section I, the goal of designing a sparse control network is inspired by cost minimization assuming that the links have a fixed cost. Therefore, the goal of reducing the cost of the communications network is equivalent to minimizing the number of links in the communications network while guaranteeing system stability.

In this section we aim to design a control network with minimum number of necessary links that satisfy general stability conditions (16) and (17). That is, we need to find observers and controllers satisfying stability conditions with minimum number of non-zero \mathbf{L}_{ij} and \mathbf{O}_{ij} . This problem can be formulated as

$$\begin{aligned} & \text{minimize} \quad \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i^{\text{in}}} 1_{\{\mathbf{L}_{ij} \neq \mathbf{0} \text{ or } \mathbf{O}_{ij} \neq \mathbf{0}\}} \\ & \text{subject to} \quad (3), (16) \text{ and } (17) \end{aligned} \quad (18)$$

for all $i, k \in \mathcal{N}$. The optimization variables in (18) are $\mathbf{K}_i, \mathbf{L}_{ij}, \mathbf{P}_i, \Delta_{ij}, \mathbf{M}_i, \mathbf{O}_{ij}, \hat{\mathbf{P}}_i$ and $\hat{\Delta}_{ij}$. Unfortunately, besides the fact that the objective function is integer valued, the first two constraints of this optimization problem are non-convex. In the following we show that this problem can be convexified by restricting its domain as follows:

Theorem 4: Consider the system in (1) with observer-based controller of the form (2) and bounds in (3). The overall network

is globally asymptotically stable if the following convex mixed-binary program has a solution

$$\begin{aligned}
& \text{minimize} \quad \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i^{\text{in}}} \alpha_{ij} \\
& \text{subject to} \quad \begin{bmatrix} \mathbf{F}_i(\mathbf{Z}_i, \mathbf{W}_i, \alpha_{ij}) & \sqrt{|\mathcal{N}_i^{\text{out}}|} \mathbf{Z}_i \\ \sqrt{|\mathcal{N}_i^{\text{out}}|} \mathbf{Z}_i & \mathbf{I}_{n_i} \end{bmatrix} \succ \mathbf{0}, \quad (\text{C1}) \\
& \quad \begin{bmatrix} \hat{\mathbf{F}}_i(\hat{\mathbf{Z}}_i, \hat{\mathbf{W}}_i, \alpha_{ji}) & \sqrt{|\mathcal{N}_i^{\text{in}}|} \hat{\mathbf{Z}}_i \\ \sqrt{|\mathcal{N}_i^{\text{in}}|} \hat{\mathbf{Z}}_i & \mathbf{I}_{n_i} \end{bmatrix} \succ \mathbf{0}, \quad (\text{C2}) \\
& \quad \mathbf{Z}_i \succ \mathbf{0}, \quad (\text{C3}) \\
& \quad \hat{\mathbf{Z}}_i \succ \mathbf{0}, \quad (\text{C4}) \\
& \quad \kappa_i \lambda_{\min}(\mathbf{Z}_i) - \sigma_{\max}(\mathbf{W}_i) \geq 0, \quad (\text{C5}) \\
& \quad \mu_i \lambda_{\min}(\hat{\mathbf{Z}}_i) - \sigma_{\max}(\hat{\mathbf{W}}_i) \geq 0, \quad (\text{C6}) \\
& \quad \alpha_{ik} \in \{0, 1\}, \quad (\text{C7})
\end{aligned} \tag{19}$$

for all $i, k \in \mathcal{N}$, where

$$\mathbf{F}_i(\mathbf{Z}_i, \mathbf{W}_i, \alpha_{ij}) \triangleq -\mathbf{Z}_i \mathbf{A}_i^T - \mathbf{A}_i \mathbf{Z}_i - \mathbf{W}_i^T \mathbf{B}_i^T - \mathbf{B}_i \mathbf{W}_i - \sum_{j \in \mathcal{N}_i^{\text{in}}} \left[\mathbf{H}_{ij} \mathbf{H}_{ij}^T + \alpha_{ij} \left(\mathbf{G}_{ij}^* \mathbf{G}_{ij}^{*T} - \mathbf{H}_{ij} \mathbf{H}_{ij}^T \right) \right], \tag{20}$$

and

$$\hat{\mathbf{F}}_i(\hat{\mathbf{Z}}_i, \hat{\mathbf{W}}_i, \alpha_{ji}) \triangleq -\mathbf{A}_i^T \hat{\mathbf{Z}}_i - \hat{\mathbf{Z}}_i \mathbf{A}_i - \mathbf{C}_i^T \hat{\mathbf{W}}_i^T - \hat{\mathbf{W}}_i \mathbf{C}_i - \sum_{j \in \mathcal{N}_i^{\text{out}}} \left[\mathbf{H}_{ji}^T \mathbf{H}_{ji} + \alpha_{ji} \left(\hat{\mathbf{G}}_{ji}^{*T} \hat{\mathbf{G}}_{ji}^* - \mathbf{H}_{ji}^T \mathbf{H}_{ji} \right) \right]. \tag{21}$$

Furthermore, if $(\mathbf{W}_i^*, \hat{\mathbf{W}}_i^*, \mathbf{Z}_i^*, \hat{\mathbf{Z}}_i^*, \alpha_{ij}^*)$ is a solution of (19), the controller and observer gains are

$$\begin{aligned}
\mathbf{K}_i &= \mathbf{W}_i^* (\mathbf{Z}_i^*)^{-1}, & \mathbf{M}_i &= (\hat{\mathbf{Z}}_i^*)^{-1} \hat{\mathbf{W}}_i^*, \\
\mathbf{L}_{ij} &= \alpha_{ij}^* \mathbf{L}_{ij}^*, & \mathbf{O}_{ij} &= \alpha_{ij}^* \mathbf{O}_{ij}^*, \\
\mathbf{G}_{ij}^* &= \mathbf{B}_i \mathbf{L}_{ij}^* + \mathbf{H}_{ij}, & \hat{\mathbf{G}}_{ij}^* &= \mathbf{O}_{ij}^* \mathbf{C}_j + \mathbf{H}_{ij}, \\
\mathbf{L}_{ij}^* &= \arg \min_{\mathbf{L}_{ij}} \|\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij}\| & \text{subject to } \|\mathbf{L}_{ij}\| &\leq \iota_{ij}, \\
\mathbf{O}_{ij}^* &= \arg \min_{\mathbf{O}_{ij}} \|\mathbf{O}_{ij} \mathbf{C}_j + \mathbf{H}_{ij}\| & \text{subject to } \|\mathbf{O}_{ij}\| &\leq \omega_{ij}.
\end{aligned} \tag{22}$$

Proof: Let us apply the congruence transformations by pre-multiplying and post-multiplying (16) by \mathbf{P}_i^{-1} , which always exists due to the positive definiteness of \mathbf{P}_i . We restrict the feasibility domain to $\Delta_{ij} \triangleq \delta \mathbf{I}_{n_i}$ where $\delta \in \mathbb{R}_{++}$. By defining new variables $\mathbf{Z}_i \triangleq \delta \mathbf{P}_i^{-1}$ and $\mathbf{W}_i \triangleq \delta \mathbf{K}_i \mathbf{P}_i^{-1}$, we can write (16) as

$$\begin{aligned}
|\mathcal{N}_i^{\text{out}}| \mathbf{Z}_i^2 + \mathbf{Z}_i \mathbf{A}_i^T + \mathbf{A}_i \mathbf{Z}_i + \mathbf{W}_i^T \mathbf{B}_i^T + \mathbf{B}_i \mathbf{W}_i + \sum_{j \in \mathcal{N}_i^{\text{in}}} (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})(\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})^T &\prec \mathbf{0}, \\
\mathbf{Z}_i &\succ \mathbf{0}.
\end{aligned} \tag{23}$$

The original variables can be obtained from these new variables: $\mathbf{P}_i = \delta \mathbf{Z}_i^{-1}$ and $\mathbf{K}_i = \mathbf{W}_i \mathbf{Z}_i^{-1}$. Note that the controller gain \mathbf{K}_i is not a function of δ . Thus, δ only scales \mathbf{P}_i , and without loss of generality, we can assume $\delta = 1$.

To satisfy (23), we can upper bound the last term by

$$(\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})(\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})^T \preceq \lambda_{\max} [(\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})(\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})^T] \mathbf{I}_{n_i}. \tag{24}$$

Thus, we can design \mathbf{L}_{ij} in (23) such that it minimizes $\lambda_{\max} [(\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})(\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})^T]$ while satisfying (3). That is

$$\begin{aligned}
\mathbf{L}_{ij}^* &= \arg \min_{\mathbf{L}_{ij}} \|\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij}\| \\
&\text{subject to } \|\mathbf{L}_{ij}\| \leq \iota_{ij}.
\end{aligned} \tag{25}$$

This suggests that our aim is to *decouple* each link \mathbf{H}_{ij} in (4) using the link $\mathbf{B}_i \mathbf{L}_{ij}$, in Euclidean norm sense, as much as possible. Obviously, if the matching condition $\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij} = \mathbf{0}$ can be satisfied, the link is completely decoupled, as stated in Corollary 2. Problem (25) is a matrix spectral norm minimization which can be written as a standard semidefinite program (see Appendix), which is readily solved.

We can now proceed to design a sparse control network. That is, we seek a set of \mathbf{L}_{ij} that guarantee asymptotic stability, with minimum number of non-zero \mathbf{L}_{ij} . Recall that when we do include the ij th link in the communications network, we should use the gain \mathbf{L}_{ij}^* . On the other hand when the ij th link is not used we have $\mathbf{L}_{ij} = \mathbf{0}$.

Define binary variables $\alpha_{ij} \in \{0, 1\}$ associated to each link \mathbf{L}_{ij} . The variable $\alpha_{ij} = 1$ if the ij th link is used in the control network and $\alpha_{ij} = 0$ if it is not used. This means that

$$\mathbf{L}_{ij} = \alpha_{ij} \mathbf{L}_{ij}^*. \quad (26)$$

To find the variables \mathbf{Z}_i and \mathbf{W}_i in (23), we use the Schur complement lemma [47] and rewrite (23) as the following feasibility problem

$$\begin{aligned} \begin{bmatrix} \mathbf{F}_i(\mathbf{Z}_i, \mathbf{W}_i, \alpha_{ij}) & \sqrt{|\mathcal{N}_i^{\text{out}}|} \mathbf{Z}_i \\ \sqrt{|\mathcal{N}_i^{\text{out}}|} \mathbf{Z}_i & \mathbf{I}_{n_i} \end{bmatrix} \succ \mathbf{0}, \\ \mathbf{Z}_i \succ \mathbf{0}, \\ \alpha_{ik} \in \{0, 1\}, \end{aligned} \quad (27)$$

for all $i, k \in \mathcal{N}$.

While changing the variables from \mathbf{P}_i and \mathbf{K}_i to \mathbf{Z}_i and \mathbf{W}_i convexified the first two constraints in (18), it caused the local gain constraint (3) to become non-convex. To remedy this, we can convexify this constraint by first upper bounding the norm of \mathbf{K}_i as

$$\begin{aligned} \|\mathbf{K}_i\| &= \|\mathbf{W}_i \mathbf{Z}_i^{-1}\| \\ &\leq \|\mathbf{W}_i\| \|\mathbf{Z}_i^{-1}\| \\ &= \sigma_{\max}(\mathbf{W}_i) \lambda_{\max}(\mathbf{Z}_i^{-1}) \\ &= \frac{\sigma_{\max}(\mathbf{W}_i)}{\lambda_{\min}(\mathbf{Z}_i)}, \end{aligned} \quad (28)$$

and forcing (3) by bounding (28) by κ_i :

$$\|\mathbf{K}_i\| \leq \frac{\sigma_{\max}(\mathbf{W}_i)}{\lambda_{\min}(\mathbf{Z}_i)} \leq \kappa_i. \quad (29)$$

Equivalently

$$\kappa_i \lambda_{\min}(\mathbf{Z}_i) - \sigma_{\max}(\mathbf{W}_i) \geq 0, \quad (30)$$

which is a convex constraint. Equations (27) and (30) together, give constraints (C1), (C3), (C5) and (C7) in (19).

Similarly, we can convexify the general stability condition (17) with given bounds in (3) as constraints (C2), (C4) and (C6) in (19) where $\hat{\Delta}_{ij} \triangleq \hat{\delta} \mathbf{I}_{n_i}$, $\hat{\delta} \in \mathbb{R}_{++}$, $\hat{\mathbf{Z}}_i \triangleq \hat{\delta} \hat{\mathbf{P}}_i$ and $\hat{\mathbf{W}}_i \triangleq \hat{\delta} \hat{\mathbf{P}}_i \mathbf{M}_i$. Note that if link ij is used we can communicate both $\mathbf{L}_{ij} \hat{\mathbf{x}}_j(t)$ and $\mathbf{O}_{ij} \mathbf{y}_j(t)$ on the same link. This means the same binary variables α_{ij} should be used for both controller and observer links. Thus, $\mathbf{O}_{ij} = \alpha_{ij} \mathbf{O}_{ij}^*$, where

$$\begin{aligned} \mathbf{O}_{ij}^* &= \arg \min_{\mathbf{O}_{ij}} \|\mathbf{O}_{ij} \mathbf{C}_j + \mathbf{H}_{ij}\| \\ \text{subject to } &\|\mathbf{O}_{ij}\| \leq \omega_{ij}. \end{aligned} \quad (31)$$

Minimizing the number of communication links is equivalent to minimizing the number of $\alpha_{ij} = 1$, or in other words, minimizing the sum of all α_{ij} subject to constraints in (19). ■

Corollary 3: Decentralized control is possible if there are no bounds on the norms of local gains, i.e. $\kappa_i = \mu_i = \infty$, and for each i we have either

- $\mathbf{B}_i \mathbf{B}_i^T$ is non-singular, or
- $|\mathcal{N}_i^{\text{out}}| \mathbf{I}_{n_i} + \mathbf{A}_i^T + \mathbf{A}_i + \sum_{j \in \mathcal{N}_i^{\text{in}}} \mathbf{H}_{ij} \mathbf{H}_{ij}^T \prec \mathbf{0}$,

and, either

- $\mathbf{C}_i^T \mathbf{C}_i$ is non-singular, or
- $|\mathcal{N}_i^{\text{in}}| \mathbf{I}_{n_i} + \mathbf{A}_i^T + \mathbf{A}_i + \sum_{j \in \mathcal{N}_i^{\text{out}}} \mathbf{H}_{ji}^T \mathbf{H}_{ji} \prec \mathbf{0}$.

Proof: Consider a decentralized solution of the form $\alpha_{ij} = 0$, $\mathbf{Z}_i = \mathbf{I}_{n_i}$ and $\mathbf{W}_i = -\beta \mathbf{B}_i^T$ where $\beta \in \mathbb{R}_{++}$. Clearly, this satisfies constraint (C3) in (19). Now, considering constraint (C1) in (19), in the form of (23), we need

$$|\mathcal{N}_i^{\text{out}}| \mathbf{I}_{n_i} + \mathbf{A}_i^T + \mathbf{A}_i - 2\beta \mathbf{B}_i \mathbf{B}_i^T + \sum_{j \in \mathcal{N}_i^{\text{in}}} \mathbf{H}_{ij} \mathbf{H}_{ij}^T \prec \mathbf{0}, \quad (32)$$

or equivalently

$$|\mathcal{N}_i^{\text{out}}| \mathbf{I}_{n_i} + \mathbf{A}_i^T + \mathbf{A}_i + \sum_{j \in \mathcal{N}_i^{\text{in}}} \mathbf{H}_{ij} \mathbf{H}_{ij}^T \prec 2\beta \mathbf{B}_i \mathbf{B}_i^T. \quad (33)$$

We can upper bound the left hand side by

$$|\mathcal{N}_i^{\text{out}}| \mathbf{I}_{n_i} + \mathbf{A}_i^T + \mathbf{A}_i + \sum_{j \in \mathcal{N}_i^{\text{in}}} \mathbf{H}_{ij} \mathbf{H}_{ij}^T \preceq |\mathcal{N}_i^{\text{out}}| \mathbf{I}_{n_i} + \lambda_{\max} \left(\mathbf{A}_i^T + \mathbf{A}_i + \sum_{j \in \mathcal{N}_i^{\text{in}}} \mathbf{H}_{ij} \mathbf{H}_{ij}^T \right) \mathbf{I}_{n_i}, \quad (34)$$

and lower bound the right hand side by

$$2\beta \lambda_{\min} (\mathbf{B}_i \mathbf{B}_i^T) \mathbf{I}_{n_i} \preceq 2\beta \mathbf{B}_i \mathbf{B}_i^T. \quad (35)$$

Thus for (33) to be satisfied, it suffices to have

$$|\mathcal{N}_i^{\text{out}}| + \lambda_{\max} \left(\mathbf{A}_i^T + \mathbf{A}_i + \sum_{j \in \mathcal{N}_i^{\text{in}}} \mathbf{H}_{ij} \mathbf{H}_{ij}^T \right) < 2\beta \lambda_{\min} (\mathbf{B}_i \mathbf{B}_i^T). \quad (36)$$

If $\mathbf{B}_i \mathbf{B}_i^T$ is non-singular, any

$$\beta > \frac{|\mathcal{N}_i^{\text{out}}| + \lambda_{\max} \left(\mathbf{A}_i^T + \mathbf{A}_i + \sum_{j \in \mathcal{N}_i^{\text{in}}} \mathbf{H}_{ij} \mathbf{H}_{ij}^T \right)}{2\lambda_{\min} (\mathbf{B}_i \mathbf{B}_i^T)}, \quad (37)$$

will satisfy (33). If $\mathbf{B}_i \mathbf{B}_i^T$ is singular, we have $\lambda_{\min} (\mathbf{B}_i \mathbf{B}_i^T) = 0$ and we need

$$|\mathcal{N}_i^{\text{out}}| + \lambda_{\max} \left(\mathbf{A}_i^T + \mathbf{A}_i + \sum_{j \in \mathcal{N}_i^{\text{in}}} \mathbf{H}_{ij} \mathbf{H}_{ij}^T \right) < 0, \quad (38)$$

or

$$|\mathcal{N}_i^{\text{out}}| \mathbf{I}_{n_i} + \mathbf{A}_i^T + \mathbf{A}_i + \sum_{j \in \mathcal{N}_i^{\text{in}}} \mathbf{H}_{ij} \mathbf{H}_{ij}^T \prec \mathbf{0}. \quad (39)$$

The argument for constraint (C2) in (19) is similar. ■

Equation (19) is a convex mixed-binary program, which in general is known to be NP-hard. In the worst case, we have a complete binary tree of depth E , where E is the total number of edges in the plant network. This means that one has to solve 2^E convex problems, carrying an exhaustive search on the binary variables. While a variety of exact methods for convex mixed-binary programs are available [48]-[50], their computational complexity is prohibitive for large networks. Here, we use a suboptimal simple relaxation approach, as follows:

- 1) Solve (19) with (C7) relaxed to $0 \leq \alpha_{ij} \leq 1$ to obtain solution $\alpha_{ij}^{(r)}$ satisfy (C1) through (C6).
- 2) Solve

$$\begin{aligned} & \underset{k,l}{\text{maximize}} \quad \tau = \alpha_{kl}^{(r)} \\ & \text{subject to} \quad \alpha_{ij} = u_{\tau}(\alpha_{ij}^{(r)}) \text{ satisfy (C1) through (C6)} \end{aligned} \quad (40)$$

where

$$u_{\tau}(\alpha_{ij}^{(r)}) = \begin{cases} 0 & \alpha_{ij}^{(r)} < \tau \\ 1 & \alpha_{ij}^{(r)} \geq \tau \end{cases} \quad (41)$$

- 3) Return $\alpha_{ij}^{\dagger} = u_{\tau^*}(\alpha_{ij}^{(r)})$, where τ^* is the solution of (40).

Note that the complexity of (40) is only $\log E$, since it can be solved by using a binary search on τ in a sorted set of $\{\alpha_{ij}^{(r)}\}$.

V. NUMERICAL EXAMPLE

This section presents a numerical example demonstrating our approach in distributed observer and controller design for distributed networked control systems with a sparse control network. The system under study (Fig. 2) is a collection of three

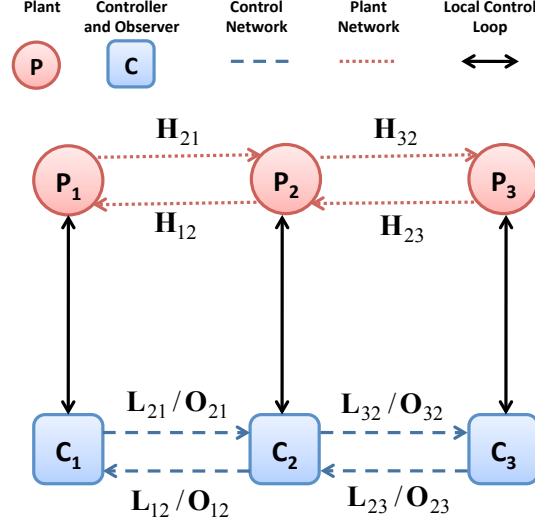


Fig. 2. Network of three coupled subsystems

LTI subsystems coupled together. We assume the following parameters:

$$\begin{aligned}
 \mathbf{A}_1 &= \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, & \mathbf{B}_1 &= \begin{bmatrix} 2 \\ 0 \end{bmatrix}, & \mathbf{C}_1^T &= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
 \mathbf{A}_2 &= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}, & \mathbf{B}_2 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \mathbf{C}_2^T &= \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \\
 \mathbf{A}_3 &= \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}, & \mathbf{B}_3 &= \begin{bmatrix} 3 \\ 0 \end{bmatrix}, & \mathbf{C}_3^T &= \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \\
 \mathbf{H}_{12} &= \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}, & \mathbf{H}_{21} &= \begin{bmatrix} 1 & -\frac{3}{4} \\ \frac{1}{2} & 1 \end{bmatrix}, & \mathbf{H}_{13} = \mathbf{H}_{31} &= \mathbf{0}, \\
 \mathbf{H}_{23} &= \begin{bmatrix} 2 & 2 \\ \frac{1}{2} & 1 \end{bmatrix}, & \mathbf{H}_{32} &= \begin{bmatrix} \frac{5}{2} & \frac{5}{2} \\ \frac{3}{4} & 1 \end{bmatrix}.
 \end{aligned} \tag{42}$$

We will consider three different cases, where we progressively increase κ_i and μ_i . Since the subsystems are controllable and observable, we can apply Theorem 4 to design distributed observers and controllers that stabilize the entire network with small number of links in the control network. First, we need to find \mathbf{L}_{ij}^* and \mathbf{O}_{ij}^* that are the solutions of norm minimization (25) and (31) respectively. We assume there are no bounds on the norms of coupling gains, i.e. $\nu_{ij} = \omega_{ij} = \infty$. The results are

$$\begin{aligned}
 \mathbf{L}_{12}^{*T} &= \begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}, & \mathbf{L}_{21}^{*T} &= \begin{bmatrix} -1 \\ \frac{3}{4} \end{bmatrix}, & \mathbf{L}_{23}^{*T} &= \begin{bmatrix} -2 \\ -2 \end{bmatrix}, & \mathbf{L}_{32}^{*T} &= \begin{bmatrix} -\frac{5}{6} \\ -\frac{5}{6} \end{bmatrix}, \\
 \mathbf{O}_{12}^* &= \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}, & \mathbf{O}_{21}^* &= \begin{bmatrix} -\frac{1}{8} \\ -\frac{3}{4} \end{bmatrix}, & \mathbf{O}_{23}^* &= \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{4} \end{bmatrix}, & \mathbf{O}_{32}^* &= \begin{bmatrix} -\frac{5}{4} \\ -\frac{7}{16} \end{bmatrix}.
 \end{aligned} \tag{43}$$

As we showed in Section IV, the controller gain \mathbf{K}_i and observer gain \mathbf{M}_i are not a function of δ and $\hat{\delta}$ respectively. Thus, without loss of generality, we set $\delta = \hat{\delta} = 1$. To solve the convex mixed-binary program (19) we will use both an exhaustive approach as well as the suboptimal algorithm provided in Section IV.

Case 1:

We start with more stringent bounds $\kappa_1 = \kappa_2 = \kappa_3 = 8$, $\mu_1 = \mu_2 = \mu_3 = 4$. Using the relaxation approach, we find $\alpha_{12}^{(r)} = 1$, $\alpha_{21}^{(r)} = 0.69$, $\alpha_{23}^{(r)} = 0.83$, $\alpha_{32}^{(r)} = 0.20$ which after thresholding yield $\alpha_{12}^\dagger = \alpha_{21}^\dagger = \alpha_{23}^\dagger = \alpha_{32}^\dagger = 1$ and

$$\begin{aligned}
 \mathbf{K}_1 &= \begin{bmatrix} -2.32 & -1.82 \end{bmatrix}, & \mathbf{K}_2 &= \begin{bmatrix} -2.86 & -0.07 \end{bmatrix}, & \mathbf{K}_3 &= \begin{bmatrix} -1.56 & 0.26 \end{bmatrix}, \\
 \mathbf{M}_1^T &= \begin{bmatrix} -3.65 & -0.09 \end{bmatrix}, & \mathbf{M}_2^T &= \begin{bmatrix} -1.10 & 0.50 \end{bmatrix}, & \mathbf{M}_3^T &= \begin{bmatrix} -1.70 & 0.35 \end{bmatrix}.
 \end{aligned} \tag{44}$$

The resulting gain norms are

$$\begin{aligned}
 \|\mathbf{K}_1\| &= 2.95, & \|\mathbf{K}_2\| &= 2.86, & \|\mathbf{K}_3\| &= 1.58, \\
 \|\mathbf{M}_1\| &= 3.65, & \|\mathbf{M}_2\| &= 1.21, & \|\mathbf{M}_3\| &= 1.74,
 \end{aligned} \tag{45}$$

which are inside the defined bounds. The gap between the norms in (45) and the bounds is due to conservatism of the bounds on controller gains and observer gains as (C5) and (C6) in (19).

Using an exhaustive search on the binary variables, followed by convex optimization of other variables, we solve convex mixed-binary program (19) to find the optimal solution $\alpha_{12}^* = \alpha_{21}^* = \alpha_{23}^* = \alpha_{32}^* = 1$, which is the same result obtained from the thresholding approach. Consequently, \mathbf{K}_i and \mathbf{M}_i are also the same.

Case 2:

Relaxing the bounds on local gains may make it possible to stabilize the network with fewer number of links. Assuming $\kappa_1 = \kappa_2 = \kappa_3 = 9$, $\mu_1 = \mu_2 = \mu_3 = 5$ and solving the convex relaxation problem, we get $\alpha_{12}^{(r)} = 0.99$, $\alpha_{21}^{(r)} = 0.23 \times 10^{-5}$, $\alpha_{23}^{(r)} = 0.80$, $\alpha_{32}^{(r)} = 0.66 \times 10^{-5}$. After thresholding, the entire network is stabilized with $\alpha_{12}^\dagger = \alpha_{23}^\dagger = 1$, $\alpha_{21}^\dagger = \alpha_{32}^\dagger = 0$ and

$$\begin{aligned} \mathbf{K}_1 &= \begin{bmatrix} -2.38 & -1.88 \end{bmatrix}, \mathbf{K}_2 = \begin{bmatrix} -4.03 & 0.77 \end{bmatrix}, \mathbf{K}_3 = \begin{bmatrix} -4.21 & -1.99 \end{bmatrix}, \\ \mathbf{M}_1^T &= \begin{bmatrix} -4.08 & -0.17 \end{bmatrix}, \mathbf{M}_2^T = \begin{bmatrix} -3.24 & -1.08 \end{bmatrix}, \mathbf{M}_3^T = \begin{bmatrix} -1.84 & 0.40 \end{bmatrix}. \end{aligned} \quad (46)$$

Norms of the local gains are

$$\begin{aligned} \|\mathbf{K}_1\| &= 3.03, \|\mathbf{K}_2\| = 4.10, \|\mathbf{K}_3\| = 4.66, \\ \|\mathbf{M}_1\| &= 4.09, \|\mathbf{M}_2\| = 3.41, \|\mathbf{M}_3\| = 1.88. \end{aligned} \quad (47)$$

We see that with these relaxed bounds, only two links are needed to stabilize the entire network. Using the exhaustive search, we find the optimal solution $\alpha_{12}^* = \alpha_{23}^* = 1$, $\alpha_{21}^* = \alpha_{32}^* = 0$ which, once again, is the same result obtained from the suboptimal approach.

Case 3:

Finally, by further relaxing the bounds to $\kappa_1 = \kappa_2 = \kappa_3 = 17$, $\mu_1 = \mu_2 = \mu_3 = 15$, even without any links the network is globally asymptotically stable. Solving the convex relaxation problem, we get $\alpha_{12}^{(r)} = \alpha_{21}^{(r)} = \alpha_{23}^{(r)} = \alpha_{32}^{(r)} = 0.30 \times 10^{-11}$, which after thresholding yield $\alpha_{12}^\dagger = \alpha_{21}^\dagger = \alpha_{23}^\dagger = \alpha_{32}^\dagger = 0$ and

$$\begin{aligned} \mathbf{K}_1 &= \begin{bmatrix} -6.05 & -8.55 \end{bmatrix}, \mathbf{K}_2 = \begin{bmatrix} -6.10 & -1.20 \end{bmatrix}, \mathbf{K}_3 = \begin{bmatrix} -4.02 & -2.08 \end{bmatrix}, \\ \mathbf{M}_1^T &= \begin{bmatrix} -4.97 & 0.00 \end{bmatrix}, \mathbf{M}_2^T = \begin{bmatrix} -5.90 & -2.39 \end{bmatrix}, \mathbf{M}_3^T = \begin{bmatrix} -3.08 & 0.34 \end{bmatrix}. \end{aligned} \quad (48)$$

Now the norms of local gains are

$$\begin{aligned} \|\mathbf{K}_1\| &= 10.47, \|\mathbf{K}_2\| = 6.21, \|\mathbf{K}_3\| = 4.53, \\ \|\mathbf{M}_1\| &= 4.97, \|\mathbf{M}_2\| = 6.36, \|\mathbf{M}_3\| = 3.10. \end{aligned} \quad (49)$$

We can see that, even though neither of conditions of Corollary 3 are satisfied in this example, as \mathbf{B}_i and \mathbf{C}_i have rank 1 and \mathbf{A}_i are unstable, this relaxation of the bounds makes a completely decentralized approach feasible. This is because of Corollary 3 is a sufficient condition for decentralization. Of course, the exhaustive search approach provides the same solution.

Case 4:

To show that relaxing the bounds does not necessarily make decentralized control feasible, we provide a counter example. Consider the system in (42) but double the coupling matrices \mathbf{H}_{ij} . Then, the convex mixed-binary problem (19) is infeasible even without constraints (C5) and (C6) in (19). Moreover, note that neither of conditions of Corollary 3 which is a sufficient condition for complete decentralization are satisfied in this example.

VI. CONCLUDING REMARKS

We have provided a design approach for distributed observer-based controllers for stabilization of networked control systems consisting of multiple coupled linear time-invariant subsystems over an arbitrary directed network. To measure states of each subsystem, we use the outputs of other nodes to improve stability of observer dynamics, in an approach dual to that of distributed controllers. Our design approach is based on a set of stability conditions obtained using the Lyapunov approach, and provides a sparse observer-controller network which guarantees global asymptotic stability. Moreover, we showed that the controller-observer network can always be a subset of the plant network. That is, it need not have any links, where one does not exist in the plant network.

Of course, inherited from the Lyapunov approach as well as assumptions made to maintain tractability, the design includes some degree of conservatism. Thus, although the results provide us with significant insight into the problem of designing the sparsest controller-observer network, a gap still remains. Quantification or reduction of this conservatism will be quite valuable.

Also of interest is the issues of convergence rate and robustness. In search of a sparse controller-observer network, we have spent the entire margin in at least one of our distributed stability criteria. This means that the resulting system has very low rate of convergence. Moreover, since there is no margin left, disturbances and model uncertainty can make the system unstable. Of course, one can add a given preset margin to each inequality. However, it is not clear how best this margin should be distributed among the inequalities to improve the convergence rate and make the network robust, without significantly growing the size of the controller-observer network. It is also interesting to understand the tradeoff between the margin and the sparsity of the observer-controller network.

We believe that the results presented in this paper provide a foothold for further progress towards understanding these interesting and important problems.

APPENDIX

FINDING \mathbf{L}_{ij}^* AND \mathbf{O}_{ij}^*

Using the fact that $\|\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij}\| \leq s_{ij}$ if and only if $(\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})(\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})^T \preceq s_{ij}^2 \mathbf{I}_{n_i}$ and $s_{ij} \in \mathbb{R}_+$, we can express the optimization problem (25) in its epigraph form

$$\begin{aligned} & \text{minimize} && s_{ij} \\ & \text{subject to} && (\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})(\mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij})^T \preceq s_{ij}^2 \mathbf{I}_{n_i}, \\ & && \mathbf{L}_{ij} \mathbf{L}_{ij}^T \preceq \iota_{ij}^2 \mathbf{I}_{m_i}, \end{aligned} \tag{50}$$

which after applying Schur complement lemma is equivalent to

$$\begin{aligned} & \text{minimize} && s_{ij} \\ & \text{subject to} && \begin{bmatrix} s_{ij} \mathbf{I}_{n_i} & \mathbf{B}_i \mathbf{L}_{ij} + \mathbf{H}_{ij} \\ \mathbf{L}_{ij}^T \mathbf{B}_i^T + \mathbf{H}_{ij}^T & s_{ij} \mathbf{I}_{n_j} \end{bmatrix} \succeq \mathbf{0}, \\ & && \begin{bmatrix} \iota_{ij} \mathbf{I}_{m_i} & \mathbf{L}_{ij} \\ \mathbf{L}_{ij}^T & \iota_{ij} \mathbf{I}_{n_j} \end{bmatrix} \succeq \mathbf{0}. \end{aligned} \tag{51}$$

This is a standard semidefinite program (SDP), and can be readily solved [46]. The argument for \mathbf{O}_{ij}^* in the optimization problem (31) is the same.

REFERENCES

- [1] S. Amin, "Smart grid: Overview, issues and opportunities. advances and challenges in sensing, modeling, simulation, optimization and control," *European Journal of Control*, vol. 17, no. 5-6, pp. 547-567, 2011.
- [2] P. Seiler and R. Sengupta, "An \mathcal{H}_∞ approach to networked control," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 356-364, 2005.
- [3] K. Lee, S. Lee, and M. Lee, "Remote fuzzy logic control of networked control system via Profibus-DP," *IEEE Transactions on Industrial Electronics*, vol. 50, no. 4, pp. 784-792, 2003.
- [4] C. Meng, T. Wang, W. Chou, S. Luan, Y. Zhang, and Z. Tian, "Remote surgery case: robot-assisted teleneurosurgery," in *Proceedings of IEEE International Conference on Robotics and Automation*, vol. 1. IEEE, 2004, pp. 819-823.
- [5] P. Ogren, E. Fiorelli, and N. Leonard, "Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment," *IEEE Transactions on Automatic Control*, vol. 49, no. 8, pp. 1292-1302, 2004.
- [6] W. Zhang, M. Branicky, and S. Phillips, "Stability of networked control systems," *IEEE Control Systems*, vol. 21, no. 1, pp. 84-99, 2001.
- [7] J. Baillieul and P. Antsaklis, "Control and communication challenges in networked real-time systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 9-28, 2007.
- [8] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138-162, 2007.
- [9] F. Wang and D. Liu, *Networked control systems: theory and applications*. Springer, 2008.
- [10] X. Wang and M. Lemmon, "Decentralized event-triggered broadcasts over networked control systems," *Hybrid Systems: computation and control*, pp. 674-677, 2008.
- [11] —, "Event-triggering in distributed networked systems with data dropouts and delays," *Hybrid systems: Computation and control*, pp. 366-380, 2009.
- [12] M. Mazo, A. Anta, and P. Tabuada, "An iss self-triggered implementation of linear controllers," *Automatica*, vol. 46, no. 8, pp. 1310-1314, 2010.
- [13] N. Sandell Jr, P. Varaiya, M. Athans, and M. Safonov, "Survey of decentralized control methods for large scale systems," *IEEE Transactions on Automatic Control*, vol. 23, no. 2, pp. 108-128, 1978.
- [14] A. Zecevic and D. Siljak, "Global low-rank enhancement of decentralized control for large-scale systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 740-744, 2005.
- [15] D. Siljak, *Decentralized control of complex systems*. Dover Publications, 2012.
- [16] X. Wang and M. Lemmon, "Event-triggered broadcasting across distributed networked control systems," in *American Control Conference, 2008*. IEEE, 2008, pp. 3139-3144.
- [17] —, "Event-triggering in distributed networked control systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 586-601, 2011.
- [18] L. Montestruque and P. Antsaklis, "Stability of model-based networked control systems with time-varying transmission times," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1562-1572, 2004.
- [19] D. Nesić and A. Teel, "Input-output stability properties of networked control systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 10, pp. 1650-1667, 2004.
- [20] G. Walsh, H. Ye, and L. Bushnell, "Stability analysis of networked control systems," *IEEE Transactions on Control Systems Technology*, vol. 10, no. 3, pp. 438-446, 2002.
- [21] M. Branicky, S. Phillips, and W. Zhang, "Stability of networked control systems: Explicit analysis of delay," in *American Control Conference, 2000*. IEEE, 2000, pp. 2352-2357.

- [22] P. Seiler and R. Sengupta, "An \mathcal{H}_∞ approach to networked control," *IEEE Transactions on Automatic Control*, vol. 50, no. 3, pp. 356–364, 2005.
- [23] N. Elia, "Remote stabilization over fading channels," *Systems and Control Letters*, vol. 54, no. 3, pp. 237–249, 2005.
- [24] M. Yu, L. Wang, T. Chu, and F. Hao, "Stabilization of networked control systems with data packet dropout and transmission delays: continuous-time case," *European Journal of Control*, vol. 11, no. 1, pp. 40–49, 2005.
- [25] J. Wu and T. Chen, "Design of networked control systems with packet dropouts," *IEEE Transactions on Automatic Control*, vol. 52, no. 7, pp. 1314–1319, 2007.
- [26] E. Witrant, C. Canudas-de Wit, D. Georges, and M. Alamir, "Remote stabilization via communication networks with a distributed control law," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1480–1485, 2007.
- [27] F. Hao and X. Zhao, "Linear matrix inequality approach to static output-feedback stabilisation of discrete-time networked control systems," *IET Control Theory and Applications*, vol. 4, no. 7, pp. 1211–1221, 2010.
- [28] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *IEEE Transactions on Automatic Control*, vol. 51, no. 2, pp. 274–286, 2006.
- [29] J. Swigart and S. Lall, "An explicit state-space solution for a decentralized two-player optimal linear-quadratic regulator," in *American Control Conference*, 2010. IEEE, 2010, pp. 6385–6390.
- [30] M. Rotkowitz and N. Martins, "On the closest quadratically invariant constraint," in *Proceedings of the 48th IEEE Conference on Decision and Control, 2009 held jointly with the 28th Chinese Control Conference, 2009*. IEEE, 2009, pp. 1607–1612.
- [31] M. Rotkowitz, R. Cogill, and S. Lall, "Convexity of optimal control over networks with delays and arbitrary topology," *International Journal of Systems, Control and Communications*, vol. 2, no. 1, pp. 30–54, 2010.
- [32] J. Swigart and S. Lall, "A graph-theoretic approach to distributed control over networks," in *Proceedings of the 48th IEEE Conference on Decision and Control, 2009 held jointly with the 28th Chinese Control Conference, 2009*. IEEE, 2009, pp. 5409–5414.
- [33] P. Parrilo, P. Shah *et al.*, "A partial order approach to decentralized control," Ph.D. dissertation, Massachusetts Institute of Technology, 2011.
- [34] P. Shah and P. Parrilo, "An optimal controller architecture for poset-causal systems," in *50th IEEE Conference on Decision and Control and European Control Conference, 2011*. IEEE, 2011, pp. 5522–5528.
- [35] —, "A partial order approach to decentralized control," in *47th IEEE Conference on Decision and Control, 2008*. IEEE, 2008, pp. 4351–4356.
- [36] J. van Schuppen, O. Boutin, P. Kempker, J. Komenda, T. Masopust, N. Pambakian, and A. Ran, "Control of distributed systems: Tutorial and overview," *European Journal of Control*, vol. 17, no. 5, pp. 579–602, 2011.
- [37] G. Barrett and S. Lafortune, "Decentralized supervisory control with communicating controllers," *IEEE Transactions on Automatic Control*, vol. 45, no. 9, pp. 1620–1638, 2000.
- [38] K. Rudie, S. Lafortune, and F. Lin, "Minimal communication in a distributed discrete-event system," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 957–975, 2003.
- [39] F. Lin, K. Rudie, and S. Lafortune, "Minimal communication for essential transitions in a distributed discrete-event system," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1495–1502, 2007.
- [40] M. Fardad, F. Lin, and M. Jovanovic, "On the optimal design of structured feedback gains for interconnected systems," in *Proceedings of the 48th IEEE Conference on Decision and Control, 2009 held jointly with the 28th Chinese Control Conference, 2009*. IEEE, 2009, pp. 978–983.
- [41] F. Lin, M. Fardad, and M. Jovanovic, "Augmented lagrangian approach to design of structured optimal state feedback gains," *IEEE Transactions on Automatic Control*, vol. 56, no. 12, pp. 2923–2929, 2011.
- [42] M. Fardad, F. Lin, and M. Jovanovic, "Sparsity-promoting optimal control for a class of distributed systems," in *American Control Conference, 2011*. IEEE, 2011, pp. 2050–2055.
- [43] M. Razeghi-Jahromi and A. Seyedi, "Stabilization of distributed networked control systems with minimal communications network," in *American Control Conference, 2011*. IEEE, 2011, pp. 515–520.
- [44] W. Haddad and V. Chellaboina, *Nonlinear dynamical systems and control: a Lyapunov-based approach*. Princeton University Press, 2011.
- [45] K. Hassan, "Nonlinear systems," *Prentice-Hall: Upper, Saddle River, NJ*, 2002.
- [46] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.
- [47] S. Boyd, L. El Ghaoul, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*. Society for Industrial Mathematics, 1994.
- [48] P. Bonami, M. Kilinc, and J. Linderoth, "Algorithms and software for convex mixed integer nonlinear programs," *Mixed Integer Nonlinear Programming*, pp. 1–39, 2012.
- [49] P. Bonami, L. Biegler, A. Conn, G. Cornuéjols, I. Grossmann, C. Laird, J. Lee, A. Lodi, F. Margot, N. Sawaya *et al.*, "An algorithmic framework for convex mixed integer nonlinear programs," *Discrete Optimization*, vol. 5, no. 2, pp. 186–204, 2008.
- [50] I. Grossmann, "Review of nonlinear mixed-integer and disjunctive programming techniques," *Optimization and Engineering*, vol. 3, no. 3, pp. 227–252, 2002.